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Logically Independent von Neumann Lattices

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Three definitions of logical independence of two von Neumann lattices $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ of two sub-von Neumann algebras $\mathcal{M}_1$, $\mathcal{M}_2$ of a von Neumann algebra $\mathcal{M}$ are given and the relations of the definitions clarified. It is shown that under weak assumptions the following notion, called "logical independence" is the strongest: $A \wedge B \neq 0$ for any $0 \neq A \in \mathcal{P}(\mathcal{M}_1)$, $0 \neq B \in \mathcal{P}(\mathcal{M}_2)$. Propositions relating logical independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ to C^*-independence, W^*-independence, and strict locality of $\mathcal{M}_1$, $\mathcal{M}_2$ are presented.

1. INTRODUCTION

Let S be a quantum system, S_1, S_2 be two subsystems of S, and assume that the observable quantities of S, S_1, S_2 are represented by (the self-adjoint parts of) a von Neumann algebra $\mathcal{M}$ and of two von Neumann subalgebras $\mathcal{M}_1$, $\mathcal{M}_2$ of $\mathcal{M}$, respectively. The quantum logics of the systems S_1, S_2, and S are then given by the corresponding projection lattices $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_1)$, and $\mathcal{P}(\mathcal{M})$. [For the operator algebraic notions used in this paper see, e.g., Takesaki (1979).] The aim of this paper is to formulate an appropriate definition of "logical independence" of the two von Neumann lattices $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_1)$, to raise the problem of characterization of logically independent pairs, and to relate logical independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_1)$ to statistical independence conditions of the pair $\mathcal{M}_1$, $\mathcal{M}_2$.

The natural logical independence condition that comes to mind is that "no nontrivial proposition in $\mathcal{P}(\mathcal{M}_1)$ should imply—or be implied by—any nontrivial proposition in $\mathcal{P}(\mathcal{M}_1)$." In Section 1 we first consider different implementations of this idea by specifying the "implies" in terms of abstract orthomodular lattices in three ways. Having the three notions of logical independence, their relation is clarified in Propositions 1–3, which show that

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under natural and weak assumptions the three definitions coincide if the lattices are distributive, and that one of the semantic independence conditions is strictly stronger than the two others in the nondistributive case. Applied to von Neumann lattices, this strongest condition, which was introduced and called simply "logical independence" in a previous paper (Rédei, 1995), requires that $A \wedge B \neq 0$ for any $0 \neq A \in \mathcal{M}_1$ and $0 \neq B \in \mathcal{M}_2$. The problem of characterizing logically independent pairs of von Neumann lattices in this sense is then raised in Section 3, and several propositions are presented that relate logical independence to other statistical independence conditions such as C^*-independence, W^*-independence, and strict locality.

The problem of logical independence of von Neumann lattices is motivated in part by the situation in quantum field theory, where a relativistically covariant net $O \to \mathcal{M}(O)$ of local von Neumann algebras $\mathcal{M}(O)$ (indexed by the open bounded spacetime regions O in the Minkowski spacetime) is given (Haag, 1992); and, if $\mathcal{M}(O_1)$ and $\mathcal{M}(O_2)$ are two von Neumann algebras associated to two spacelike-separated spaetime regions O_1, O_2, then the two local systems confined to the regions O_1, O_2 cannot influence each other, i.e., the two systems should be independent. Indeed, in this framework there are a number of nonequivalent notions of "independence" of the local algebras [see Summers (1990) for a review], and one expects that the independence of the observable algebras is also manifest in the corresponding quantum logics being logically independent. It will be seen that this is indeed the case.

2. THREE NOTIONS OF LOGICAL INDEPENDENCE

Let L be an orthomodular lattice with minimal and maximal elements $0, I$ and with lattice operations $\wedge, \vee, \perp$, and let L_1 and L_2 be two orthomodular sublattices. Let $A \models B$ denote a semantic entailment relation between the propositions A, B. The idea of semantic independence of L_1, L_2 is that no (nontrivial) proposition in L_1 should imply any (nontrivial) proposition in L_2 and conversely, no (nontrivial) proposition in L_2 should imply any (nontrivial) proposition in L_1, where "implies" is meant in the sense of semantic entailment:

Definition 1. L_1, L_2 are *semantically independent* if

$$\text{semantic independence;} \qquad A \not\models B, \quad B \not\models A \tag{1}$$

for any $0, I \neq A \in L_1$ and $0, I \neq B \in L_2$.

Assume now that there exists a two-place implication connective $\Rightarrow$ in L representing certain features of the "implication" understood as semantic entailment. The idea of $\Rightarrow$-independence of L_1, L_2 is that no (nontrivial) proposition in L_1 "implies" or is implied by any (nontrivial) proposition in

L_2, where "implies" is now taken in the sense that the inference (with respect to $\Rightarrow$) between the elements of L_1 and L_2 is not a tautology:

Definition 2. L_1, L_2 are called *$\Rightarrow$-independent* if

$$\Rightarrow\text{-independence:} \quad (A \Rightarrow B) \neq I \quad \text{and} \quad (B \Rightarrow A) \neq I \tag{2}$$

for any $0, I \neq A \in L_1$ and $0, I \neq B \in L_2$.

In a previous paper (Rédei, 1994) the following definition of logical independence was formulated:

Definition 3. L_1, L_2 are *logically independent* if

$$\text{logical independence:} \quad (A \wedge B) \neq 0 \tag{3}$$

for any $0 \neq A \in L_1$, $0 \neq B \in L_2$.

Definition 3 also expresses semantic independence in the following sense: If $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a set of closed formulas (sentences) and Φ is a closed formula in the language $\mathscr{L}$, then Φ is said to be implied by (to be a consequence of) Γ (notation: $\Gamma \vDash_s \Phi$), if the class of sentences $\Gamma \cup \{\neg\Phi\}$ cannot be satisfied, where $\neg\Phi$ denotes the negation of the formula Φ and the subscript s in $\vDash_s$ indicates that this semantic entailment is different from the one in Definition 1. If $\mathscr{L}$ is just the language of the classical propositional calculus, then $\Gamma \vDash_s \Phi$ is equivalent to the condition that the representative in the propositional algebra of the sentence $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \wedge (\neg\Phi)$ is equal to the empty set, i.e., it is the zero element 0 of the propositional algebra, which is a Boolean algebra (orthocomplemented distributive lattice). Based on this notion of dependence, Φ can be said to be *logically independent* of Γ if neither Φ nor $(\neg\Phi)$ is implied by Γ: $\Gamma \nvDash_s \Phi$, $\Gamma \nvDash_s (\neg\Phi)$, i.e., if both $\Gamma \cup \{\Phi\}$ and $\Gamma \cup \{\neg\Phi\}$ can be satisfied. This logical independence corresponds to the algebraic condition that the representatives in the propositional algebra of neither of the sentences $(\gamma_1 \wedge \gamma_2 \cdots \wedge (\neg\Phi))$ and $(\gamma_1 \wedge \gamma_2 \cdots \wedge \Phi)$ is equal to the zero element. Generalizing, one can define two sets of sentences Γ_1 and Γ_2 to be logically independent of each other if each of the four sets of sentences $\Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cup \neg\Gamma_2$, $\neg\Gamma_1 \cup \Gamma_2$, and $\neg\Gamma_1 \cup \neg\Gamma_2$ is a satisfiable set of sentences ($\neg\Gamma$ denoting the set of sentences containing the sentences $\neg\gamma_i$, $\gamma_i \in \Gamma$). This leads to the following notion of logical independence of two Boolean subalgebras $\mathscr{B}_1$ and $\mathscr{B}_2$ of a Boolean algebra $\mathscr{B}$: $\mathscr{B}_1$ and $\mathscr{B}_2$ are said to be logically independent if for any nonzero $A_1 \in \mathscr{B}_1$ and for any nonzero $A_2 \in \mathscr{B}_2$ their meet is nonzero: $A_1 \wedge A_2 \neq 0$. This definition does not depend on the distributivity of the lattices $\mathscr{B}$, $\mathscr{B}_1$, and $\mathscr{B}_2$, and so the definition can be immediately carried over to the case where $\mathscr{B}$ is replaced by an orthomodular lattice.

Having the three definitions of independence, we ask the question of the relation between them. Clearly, as long as the key elements in the definitions (the semantic entailment $\models$ in Definition 1, the connective $\Rightarrow$ in Definition 2, and the lattice operation $\wedge$ in Definition 3) are not linked to each other, the three independence notions remain unrelated.

In the case of the so-called concrete quantum logic, where L is $\mathcal{P}(\mathcal{H})$, the lattice of projections of a Hilbert space $\mathcal{H}$, the semantic entailment $\models$ is identified with the usual partial ordering $\leq$ in $\mathcal{P}(\mathcal{H})$: $A \models B$ if and only if $A \leq B$. Now, if $\models$ is identified with $\leq$ in the orthomodular lattice L, then if $A \leq B$ for some $A \in L_1$, $B \in L_2$, then A is orthogonal to $B^\perp$; consequently $A \wedge B^\perp = 0$, and since $B^\perp$ is in L_2 if B is (L_2 is a sublattice), L_1, L_2 are not logically independent. So we have the following result.

Proposition 1. If the semantic entailment $\models$ is given by the partial ordering in L, then logical independence of L_1, L_2 implies that L_1, L_2 are semantically independent.

The connective $\Rightarrow$, too, is related to the semantic entailment: The standard way of specifying $\Rightarrow$ is in terms of the so-called "implicative criteria" that link $\Rightarrow$ to $\models$. What one expects in this direction is that the more faithfully $\Rightarrow$ reflects the properties of the semantic entailment, the closer the notions of semantic and $\Rightarrow$-independence are. Let us recall the minimal implicative criteria.

E If $A \models B$, then $(A \Rightarrow B) = I$
MP $A \wedge (A \Rightarrow B) \models B$
MT $\sim B \wedge (A \Rightarrow B) \models \sim A$

E (law of entailment), **MP** (modus ponens), and **MT** (modus tollens) are called minimal implicative criteria; every reasonable implication connective is supposed to satisfy them (Hardegree, 1976, 1979).

Assume now that the semantic entailment $\models$ is given by the partial ordering $\leq$ in L. Then **E** and **MP** together are equivalent to

$$A \leq B \quad \text{if and only if} \quad (A \Rightarrow B) = I \tag{4}$$

Obviously, the following proposition is true then:

Proposition 2. If the semantic entailment is given by the partial ordering in the orthomodular lattice L and $\Rightarrow$ is an implication connective satisfying the minimal implicative criteria, then the semantic independence of L_1, L_2 is equivalent to $\Rightarrow$-independence of L_1, L_2, and logical independence implies both the semantic and $\Rightarrow$-independence.

We note at this point that in an orthomodular lattice there exist implication connectives satisfying the minimal implicative criteria. Examples are the so-called quantum conditionals: In any orthomodular lattice there exist three conditional operations which can be written as lattice polinoms and which satisfy **E**, **MP**, and **MT**. These three conditionals are the following (Hardegree, 1976, 1979):

$$A \Rightarrow_1 B = A^{\perp} \vee (A \wedge B)$$

$$A \Rightarrow_2 B = (A^{\perp} \wedge B^{\perp}) \vee B$$

$$A \Rightarrow_3 B = (A \wedge B) \vee (A^{\perp} \wedge B) \vee (A^{\perp} \wedge B^{\perp})$$

Each of these conditionals is a generalization of the classical material implication $(A \Rightarrow B) = A^{\perp} \vee B$ in the sense that each of the three reduces to the classical conditional if restricted to a Boolean sublogic.

The next proposition gives a sufficient condition for L_1, L_2 which implies that the $\Rightarrow$-independence of L_1, L_2 implies logical independence.

Proposition 3. Let $\models$ be given by $\leq$ and assume that $\Rightarrow$ satisfies the minimal implicative criteria. If L_1, L_2 are such that the orthomodular lattice generated by any two elements $A \in L_1$, $B \in L_2$ is a distributive sublattice in L, then $\Rightarrow$-independence of L_1, L_2 implies logical independence of L_1, L_2.

Proof. Assume that L_1, L_2 are not logically independent. Then there exist $0 \neq A \in L_1$, $0 \neq B \in L_2$ such that $A \wedge B = 0$. Using the distributivity assumption, we can write then

$$\begin{aligned} A &= A \wedge I = A \wedge (B \vee B^{\perp}) \\ &= (A \wedge B) \vee (A \wedge B^{\perp}) = 0 \vee (A \wedge B^{\perp}) = A \wedge B^{\perp} \end{aligned}$$

which implies $A \leq B^{\perp}$ and so by (4) we have $(A \Rightarrow B^{\perp}) = I$, and L_1, L_2 are not $\Rightarrow$-independent.

In the projection lattice of a von Neumann algebra a sublattice generated by a set Z of projections is distributive if and only if the projections in Z are pairwise commuting; hence as a particular case of Proposition 3 we have the following result.

Proposition 4. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be mutually commuting von Neumann subalgebras of the von Neumann algebra $\mathcal{M}$ (i.e., $AB = BA$ for every $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$) and $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ be the corresponding von Neumann lattices. If $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ are $\Rightarrow$-independent with respect to any of the quantum conditionals $\Rightarrow$ (which in this case means that the lattices are actually independent with respect to the classical material implication), then $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ are logically independent.

Thus we see that in the case when L is distributive and L_1, L_2 are two distributive sublattices, the three independence notions coincide under the natural identification of $\models$ with $\leq$, and under the assumption that $\Rightarrow$ satisfies minimal implicative criteria. However, in the nondistributive, truly quantum case logical independence is strictly stronger than semantic or $\Rightarrow$-independence, since $\Rightarrow$-independence does not imply logical independence, as the following counterexample shows:

Consider the six-element orthomodular lattice

$$L_6 = \{A, A^\perp, B, B^\perp, 0, I\}$$

with the partial ordering given by

$$0 \leq X \leq I \qquad (X = A, A^\perp, B, B^\perp)$$

If $\models$ is taken to be $\leq$ and $\Rightarrow$ satisfies the minimal implicative criteria, then the two sublattices $L_1 = \{0, A, A^\perp, I\}$ and $L_2 = \{0, B, B^\perp, I\}$ are $\Rightarrow$-independent (and also semantically independent), but not logically independent.

3. LOGICAL AND STATISTICAL INDEPENDENCE

Now the question is: do logically independent pairs of von Neumann lattices exist? The answer is yes. One way to characterize logical independence is to relate logical independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ to statistical independence conditions formulated for the algebras $\mathcal{M}_1$, $\mathcal{M}_2$. Let us first recall the relevant definitions of statistical independence.

Two C^*-subalgebras $\mathcal{A}_1$, $\mathcal{A}_2$ of the C^*-algebra $\mathcal{C}$ are called C^*-independent if for any state ϕ_1 on $\mathcal{A}_1$ and for any state ϕ_2 on $\mathcal{A}_2$ there is a state ϕ on $\mathcal{C}$ such that $\phi(A) = \phi_1(A)$ and $\phi(B) = \phi_2(B)$ ($A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$) (Haag and Kastler, 1964; Roos, 1970). The C^*-independence of the pair $(\mathcal{A}_1, \mathcal{A}_2)$ means that no preparation in any state of the system described by $\mathcal{A}_1$ can exclude any preparation of the system described by $\mathcal{A}_2$. The pair $(\mathcal{M}_1, \mathcal{M}_2)$ is said to be W^*-independent if for any *normal* state ϕ_1 on $\mathcal{M}_1$ and for any *normal* state ϕ_2 on $\mathcal{M}_2$ there is a *normal* state ϕ on $\mathcal{M}$ that extends both ϕ_1 and ϕ_2 (Summers, 1990). Let $(\mathcal{M}_1, \mathcal{M}_2)$ be an (ordered) pair of von Neumann subalgebras of the von Neumann algebra $\mathcal{M}$. The pair $(\mathcal{M}_1, \mathcal{M}_2)$ is said to have the independence condition strict locality (or said to be strictly local) if for any $B \in \mathcal{P}(\mathcal{M}_2)$ and for any normal state ϕ_1 on $\mathcal{M}_1$ there exists a normal state ϕ on $\mathcal{M}$ such that $\phi(B) = 1$ and $\phi(A) = \phi_1(A)$ for all $A \in \mathcal{M}_1$ (Kraus, 1964). If $\mathcal{M}_1$, $\mathcal{M}_2$ are mutually commuting, then W^*-independence of $(\mathcal{M}_1, \mathcal{M}_2)$ implies strict locality of the pair, and strict locality implies C^*-independence of $(\mathcal{M}_1, \mathcal{M}_2)$ [for proofs of these statements see Summers (1990)].

Proposition 5. The following statements hold:

(i) If $\mathcal{M}_1$, $\mathcal{M}_2$ are mutually commuting, then $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ are logically independent if and only if $\mathcal{M}_1$, $\mathcal{M}_2$ are C^*-independent.

(ii) If $\mathcal{M}$ is a finite-dimensional full matrix algebra, then C^*-independence of $\mathcal{M}_1$, $\mathcal{M}_2$ implies logical independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$, whether or not $\mathcal{M}_1$, $\mathcal{M}_2$ are mutually commuting.

(iii) If $(\mathcal{M}_1, \mathcal{M}_2)$ is either a W^*-independent or a strictly local pair, then $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ are logically independent whether or not $\mathcal{M}_1$, $\mathcal{M}_2$ are mutually commuting.

For the proofs of the statements in the above proposition as well as for some further statements, and for an example of a logically nonindependent pair of von Neumann lattices we refer to Rédei (1995).

As a consequence of (i) in Proposition 5, the logics $\mathcal{P}(\mathcal{M}(O_1))$ and $\mathcal{P}(\mathcal{M}(O_2))$ associated to the von Neumann algebras $\mathcal{M}(O_1)$, $\mathcal{M}(O_2)$ of local observables localized in spacelike-separated wedge and double cone regions O_1, O_2 are logically independent, since the algebras $\mathcal{M}(O_1)$, $\mathcal{M}(O_2)$ commute by the axiom of microcausality and they also are C^*-independent (Haag, 1992).

A simpler example of a mutually commuting C^*-independent pair of C^*-algebras is the pair $(M_n \otimes I, I \otimes M_n)$, where M_n is the full matrix algebra of complex n by n matrices and $M = M_n \otimes M_n$. The lattices $\mathcal{P}(M_n \otimes I)$, $\mathcal{P}(I \otimes M_n)$ are then logically independent. A special case is the Bohm–Bell system $M_n \otimes M_2$, so the propositions on the events at the "left and right wings" of the system are logically independent. Note that there are also examples of finite-dimensional matrix algebras that do not mutually commute and are C^*- (hence also W^*-) independent (Summers, 1990). The corresponding lattices are logically independent by (iii) of Proposition 5.

Proposition 5 does not give a complete characterization of logical independence—not even in regard to statistical independence. Let us formulate just two open questions:

Problem 1. Does C^*-independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ imply logical independence in general, i.e., when $\mathcal{M}_1$, $\mathcal{M}_2$ are not mutually commuting and $\mathcal{M}$ is arbitrary? In view of (ii) in Proposition 5 one should first investigate the case when $\mathcal{M}$ is a nondiscrete, finite von Neumann algebra.

Problem 2. What conditions in addition to logical independence of $\mathcal{P}(\mathcal{M}_1)$, $\mathcal{P}(\mathcal{M}_2)$ imply mutual commutativity of $\mathcal{M}_1$, $\mathcal{M}_2$?

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